Kardar-Parisi-Zhang equation with temporally correlated noise: A self-consistent approach

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In this paper we discuss the well known Kardar-Parisi-Zhang (KPZ) equation driven by temporally correlated noise. We use a self-consistent approach to derive the scaling exponents of this system. We also draw general conclusions about the behavior of the dynamic structure factor $\Phi_q(t)$ as a function of time. The approach we use here generalizes the well known self-consistent expansion (SCE) that was used successfully in the case of the KPZ equation driven by white noise, but unlike SCE, it is not based on a Fokker-Planck form of the KPZ equation, but rather on its Langevin form. A comparison to two other analytical methods, as well as to the only numerical study of this problem is made, and a need for an updated extensive numerical study is identified. We also show that a generalization of this method to any spatiotemporal correlations in the noise is possible, and two examples of this kind are considered.

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I. INTRODUCTION

Nonequilibrium surface growth processes often exhibit a phenomenon called kinetic roughening, where the surface develops a self-affine morphology [1]. Much attention has been given to a special class of models (ballistic deposition, Eden, or polynucleation growth), which are described by the Kardar-Parisi-Zhang (KPZ) equation [2]

$$\frac{\partial h(\vec{r},t)}{\partial t} = \nu\nabla^2 h + \frac{\lambda}{2}(\boldsymbol{\nabla} h)^2 + \eta(\vec{r},t), \qquad (1)$$

where $h(\vec{r},t)$ is the local height of the surface above a d-dimensional substrate in a $(d+1)$ -dimensional space, λ characterizes the tilt dependence of the growth velocity, ν is an effective surface tension, and $\eta(\vec{r},t)$ is a noise term.

Solutions of Eq. (1) exhibit scaling behavior. The simplest quantity to investigate is the surface width $W(L,t)$ that scales as (see Ref. [3])

$$W(L,t) = \frac{1}{\sqrt{L}}\left\langle \sum_{\vec{r}} [h(\vec{r},t) - \overline{h}(t)]^2 \right\rangle^{1/2} = L^\alpha g\left(\frac{t}{L^z}\right), \qquad (2)$$

where $\overline{h}(t)$ is the mean height of the interface at time t, α is the roughness exponent of the interface, and z is the dynamic exponent that describes the scaling of the relaxation time with L—which is the size of the system. The bracket $\langle\cdots\rangle$ denotes noise averaging. The scaling function $g(u)$ behaves like $g(u)\sim u^\beta$ (where β is the growth exponent) for small u's (i.e., for $t \ll L^z$) and like a constant [i.e., $g(u)\sim\text{const}$] for large u's (i.e., for $t \gg L^z$). It is easily verified from Eq. (2) that $\beta = z/\alpha$. The scaling exponents α and z describe the asymptotic behavior of the growing interface in the hydrodynamic limit.

The KPZ equation with uncorrelated noise has been well studied. For the one-dimensional case one can easily obtain exact results of $\alpha=1/2$ and $z=3/2$ by mapping the KPZ equation into the Burgers equation [2] or by using the Fokker-Planck equation associated with the Langevin form given by Eq. (1) [1]. However, for higher dimensions $(d>1)$ there are no exact results and the critical exponents have been evaluated numerically or obtained using various analytical methods (for a review see Refs. [1,4]).

The noise in the KPZ equation is a result of a physical process. As such it must be correlated in space and in time. If the correlations in space and time are short ranged it may be expected that the long distance and the long-time behavior of the system characterized by the exponents α and z are those obtained in the case of uncorrelated noise. There may be, however, situations in which the decay of correlations in the noise is algebraic.

Indeed, in some experimental situations the measured scaling exponents are larger than the values predicted by KPZ [1,4]. A possible explanation of such a departure from KPZ behavior may be long-range correlations in the noise. Such experimental results serve as a motivation for the study of systems with correlated noise in spite of the fact that direct evidence for long-range correlations in the noise is usually lacking.

Many studies of growth models with noise that is algebraically correlated in space but uncorrelated in time described by

$$\langle \eta(\vec{r},t)\rangle = 0 \qquad (3)$$

and

$$\langle \eta(\vec{r},t)\eta(\vec{r}',t)\rangle = 2D_0|\vec{r}-\vec{r}'|^{2\rho-d}\delta(t-t') \qquad (4)$$

have been published in the last decade. These include discrete one-dimensional models [ballistic deposition (BD) [5–7], solid-on-solid (SOS) [7,8], and direct (discrete) integration of the KPZ equation [5]]. Many researchers studied the KPZ equation with such noise [9–16] and obtained different predictions. In spite of the differences in the predicted

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values of the critical exponents, a common picture seems to result from all methods, namely: for small ρ's the critical exponents are the same as for the case of uncorrelated noise. Then, for ρ's above a certain critical value ρ_c the exponents become ρ dependent.

In sharp contrast to the variety of numerical results and theoretical predictions for the critical exponents of the KPZ equation with spatially correlated noise, only few results are available for the KPZ equation with temporally correlated noise — not to mention noise that is both spatially and temporally correlated. Similar to Eqs. (3) and (4), temporally correlated noise with zero mean can be described by

$$\langle \eta(\vec{r},t)\eta(\vec{r}',t)\rangle = 2D_0(\vec{r}-\vec{r}')|t-t'|^{2\phi-1}, \tag{5}$$

where ϕ characterizes the decay of the correlations over time (it is assumed that $\phi < 1/2$ or otherwise the correlations does not decay, but rather increases with time).

The first theoretical prediction of the critical exponents of KPZ in the presence of this type of noise is due to Medina *et al.* [13] that used dynamic renormalization group (DRG) analysis to study this problem. They solved the DRG equations numerically in one dimension, for the case where D_0 is a short-range function, and found out, just like for spatially correlated noise, that for small enough ϕ's the correlations are irrelevant. They claim that for $\phi > 0.167$ the correlations become relevant, and the roughness exponent can be fitted numerically by

$$\alpha_{DRG}(\phi) = 1.69\phi + 0.22. \tag{6}$$

The dynamic exponent can then be obtained using the scaling relation

$$z_{DRG}(\phi) = \frac{2\alpha_{DRG}(\phi)+1}{1+2\phi}. \tag{7}$$

These predictions have been checked numerically by Lam *et al.* [17] using the ballistic deposition model. They found sensible agreement between the DRG prediction and the numerical values they obtained. However, substantial deviations were found, centered around the expected threshold point $\phi_0 = 0.167$. Thus, the authors believe that these discrepancies are due to a crossover effect in the simulation and not due to any approximation in the DRG calculation.

Apart from the above mentioned DRG result, there is only one more result for KPZ with temporally correlated noise proposed by Ma and Ma [18] who used a Flory-like scaling approach (SA), originally suggested in the white-noise KPZ context, by Hentschel and Family [13]. Ma and Ma obtained the following strong-coupling roughness exponent:

$$\alpha_{SA}(\phi) = \frac{2+4\phi}{2\phi+d+3}, \tag{8}$$

and the following dynamic exponent

$$z_{SA}(\phi) = \frac{2d+4}{2\phi+d+3}. \tag{9}$$

These values are said to describe the strong-coupling scaling exponents for all values of the parameter ϕ, and for every dimension d. Actually, it is easily verified that these expressions reduce to the well-known white-noise KPZ results in one dimension when $\phi = 0$.

This prediction for the critical exponents is obviously different from the previous one-dimensional DRG result in two respects. First, Ma and Ma do not predict that for small enough ϕ's the temporal correlations are irrelevant, so obviously they rule out the threshold value of ϕ, ϕ_0. Second, for $\phi > 0.167$ the two approaches yield different numerical values for the scaling exponents.

This situation, where only two theoretical predictions are available for the KPZ problem in the presence of temporally correlated noise, especially when one of them (DRG) is a one-dimensional result, certainly calls for a clarification of this issue. This problem is further complicated by the fact that only one numerical study [17], and only in one dimension, is available.

At this point it is interesting to mention another result for the KPZ equation in the presence of noise with special mixed spatiotemporal correlations [nonseparable noise correlator $D(q,\omega)$]. This is a case where in contrast to systems where the noise is only suspected to be of long range, here long-range correlations in the noise follow from direct physical arguments. This problem has been studied both numerically and analytically by Li *et al.* [19] with good agreement between the analytical and numerical values. Since we deal with this problem in Sec. VI we will not discuss it further now.

In this paper we develop a self-consistent approach to deal with nonlinear Langevin equations, such as KPZ, with temporally correlated noise. Actually, as will be seen in Sec. VI, this approach can be easily generalized to spatiotemporally correlated noise. We begin with a brief derivation of the scaling exponents of the linear theory (also known as the Edwards-Wilkinson equation) in the presence of temporally correlated noise. Then, the full time-dependent two-point function for the linear problem is derived. This result will serve as a reference for the more general nonlinear discussion. In Sec. III concepts emanating from a previous self-consistent Fokker-Planck expansion to the KPZ equation are reviewed. In Sec. IV the time-dependant self-consistent approach is established. It is shown that analysis of the time-dependant self-consistent equation in the limit of short times and long times yields two static equations that are an interesting generalization of the former self-consistent Fokker-Planck expansion.

In Sec. V a detailed asymptotic solution of the self-consistent equations is obtained. In this section, we derive the different possible phases and their corresponding scaling exponents. Special attention is given to the results in one dimension. Section VI generalizes the previous results to the case of noise with arbitrary spatiotemporal correlations, and two elaborated examples are given. At the end, in Sec. VII a brief summary of the results obtained in this paper is presented.

II. THE LINEAR THEORY: THE EDWARDS-WILKINSON EQUATION

At the beginning of this paper we would like to discuss first the linear theory (i.e., the KPZ equation with its coupling constant set to zero, $\lambda=0$), namely the Edwards-Wilkinson (EW) equation [20], with temporally correlated noise. The Edwards-Wilkinson equation is

$$\frac{\partial h}{\partial t}(\vec{r},t) = \nu\nabla^2 h + \eta(\vec{r},t). \tag{10}$$

As mentioned above, in this paper we discuss temporally correlated noise characterized by

$$\langle \eta(\vec{r},t)\eta(\vec{r}',t')\rangle = 2D_0\delta(\vec{r}-\vec{r}')|t-t'|^{2\phi-1}, \tag{11}$$

where the case of uncorrelated noise corresponds to the limit $\phi=0$.

The interface that grows under these conditions is known to be self-affine, which means that if the spatial coordinates are scaled by a factor of b (i.e., $\vec{r}\to\vec{r}'=b\vec{r}$) then if we perform the transformations $t\to t'=b^z t$ and $h\to h'=b^\alpha h$ (with the appropriate scaling exponents α—the roughness exponent, and z—the dynamic exponent) as well, the statistical properties of the surface are left invariant. Since the growth equation (10) is linear, following Ref. [1] it is possible to extract the scaling exponents by scaling $\vec{r},t$, and h in the equation according to the above-mentioned transformation. But first, we have to realize that under this transformation the noise term scales like $\eta\to\eta'=b^{[z(2\phi-1)-d]/2}\eta$ (see Ref. [1]). Using this we can plug it back into the EW equation and we get

$$b^{\alpha-z}\frac{\partial h'}{\partial t'}(\vec{r}',t') = b^{\alpha-2}\nu\nabla'^2 h' + b^{[z(2\phi-1)-d]/2}\eta'(\vec{r}',t'). \tag{12}$$

Now, imposing the requirement that Eq. (10) remains invariant under this scaling transformation, namely, requiring that both equations [Eqs. (10) and (12)] should be exactly the same, we get

$$z=2 \quad \text{and} \quad \alpha=(4\phi+2-d)/2. \tag{13}$$

(This gives the roughness exponent as long as the resulting α is positive, otherwise the surface is flat). It is easily seen that this result reduces to the standard EW exponents (i.e., for the EW equation with uncorrelated noise) in the limit of $\phi=0$.

This simple result shows that temporally correlated noise tends to make the surface rougher (a bigger roughness exponent α implies a rougher surface).

The information extracted so far regrading the EW equation in the presence of temporally correlated noise could have been satisfactory. However, because we are interested in obtaining the exponents of the nonlinear theory as well, we would like to gain as much insight into the behavior of the linear problem, so that it might help us when dealing with the KPZ nonlinearity. For example, because of the linear character of Eq. (10), we can obtain the scaling form (and recover the exponents) by solving the growth equation exactly. Fourier transforming, Eq. (10), in space and time we obtain

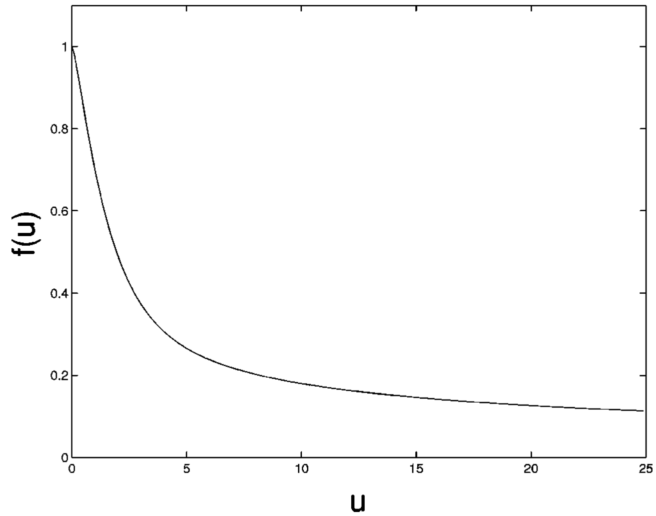

FIG. 1. The scaling function $f_{EW}(u)$ ($\phi=1/4$ was taken for this illustration). One can see an exponential-like decay for small u's, and a power-law decay for large u's.

$$h_{q\omega} = \frac{\eta_{q\omega}}{i\omega+\nu q^2}, \tag{14}$$

where $\eta_{q\omega}$ is the Fourier transform of $\eta(\vec{r},t)$. Thus, using the Fourier transform of Eq. (11), we obtain the dynamical structure factor (or the two-point correlation function)

$$\Phi_{q\omega} = \langle h_{q\omega}h_{-q,-\omega}\rangle = 2D_0\frac{\omega^{-2\phi}}{\omega^2+\nu^2q^4}. \tag{15}$$

By Fourier transforming back we get

$$\Phi_q(t) = \langle h_q(0)h_{-q}(t)\rangle = \frac{D_0}{\nu^{1+2\phi}\cos(\pi\phi)}q^{-2-4\phi}f_{EW}(\nu q^2 t). \tag{16}$$

Here $f_{EW}(u)$ is a scaling function that can be written explicitly as

$$f_{EW}(u) = \frac{\cos(\pi\phi)}{\pi}\int_{-\infty}^{\infty}\frac{y^{-2\phi}}{y^2+1}e^{iyu}dy = \cosh(u) - \frac{u^{1+2\phi}}{\Gamma(2+2\phi)}\,{}_1F_2\left(\begin{array}{c}1\\ \phi+1,\phi+\frac{3}{2}\end{array}\middle|\frac{u^2}{4}\right). \tag{17}$$

where $\Gamma(x)$ is just Euler's Gamma function, and ${}_1F_2$ is a generalized hypergeometric function. The function $f_{EW}(u)$ is also plotted in Fig. 1. As can be seen in the figure, the scaling function behaves like a constant for small u's (this corresponds to short times, that is for $\nu q^2 t \ll 1$). At the other extreme, i.e., for large u's, this function decays algebraically. In order to be sure of this power-law tail, and to obtain its exact shape we calculated the leading behaviors for small and large u's and obtained

$$f_{EW}(u) \sim \begin{cases} 1 - \dfrac{1}{\Gamma(2\phi+2)}u^{1+2\phi} + \cdots, & u \ll 1 \\ \dfrac{u^{2\phi-1}}{\Gamma(2\phi)}\left(1 + \dfrac{4(1-\phi)\left(\frac{1}{2}-\phi\right)}{u^2} + \cdots\right), & u \gg 1 \end{cases} \tag{18}$$

so that $f_{EW}(u) \sim u^{-(1-2\phi)}$ for large u's.

Naturally, the scaling exponents can be recovered easily from $\Phi_q(t)$. Since $\Phi_q(t)$ depends on time only through the combination $\nu q^2 t$ we identify the dynamic exponent as the power of q in this scaling form, so that here $z=2$. In addition, it can be seen that for small q's, $\Phi_q(t) \sim q^{-2-4\phi}$. Thus we identify the exponent $\Gamma=2+4\phi$ that can be translated into the roughness exponent via the relation $\alpha=(\Gamma-d)/2$ [see Eq. (46) below], so that we recover $\alpha=(4\phi+2-d)/2$.

The results obtained in this section will serve us later. First, it might be interesting to compare these results with the results obtained for the nonlinear theory (for example, in the weak-coupling regime of the KPZ equation). Second, we will use the scaling function of the linear theory as an ansatz for the integral equation that will determine the scaling exponents of the strong-coupling phase of the nonlinear theory.

III. NONLINEAR THEORY: THE KPZ EQUATION

We proceed now to the much harder nonlinear case that poses many technical difficulties already in the uncorrelated case.

The method we present in the following section is based on the same general ideas as the self-consistent expansion (SCE) used for systems with noise that is uncorrelated in time [9,25,26]. Namely, an expansion around an optimal linear system. The SCE is based on constructing a Fokker-Planck equation for the probability distribution of the height function. This step is based on the fact that the noise is not correlated in time. The self-consistent expansion is formulated in terms of the steady-state structure factor (or two-point function), $\phi_q=\langle h_q h_{-q}\rangle_S$ and its corresponding steady-state decay rate that describes the rate of decay of a disturbance of wave vector $\vec{q}$ in steady state, namely

$$\omega_q^{-1} = \frac{\int_0^\infty \langle h_q(t)h_{-q}(0)\rangle dt}{\langle h_q h_{-q}\rangle_S}. \tag{19}$$

The linear model around which the expansion is constructed is chosen to yield the (unknown) ϕ_q and ω_q that appear in it as parameters. An evaluation of ϕ_q and ω_q as an expansion around that linear model leads to the coupled equations

$$\phi_q = \phi_q + c_q\{\phi_p,\omega_p\}, \tag{20}$$

and

$$\omega_q = \omega_q + d_q\{\phi_p,\omega_p\}. \tag{21}$$

Within this framework, the structure factor and decay rate are obtained by solving the coupled nonlinear integral equations $c_q\{\phi_p,\omega_p\}=0$ and $d_q\{\phi_p,\omega_p\}=0$. In contrast to other expansions, the full correction, in a given order of the expansion, of the relevant physical quantities, is really small. In fact, it is chosen to be zero.

IV. DERIVATION OF THE TIME-DEPENDENT SELF-CONSISTENT APPROACH

In this work (following Ref. [22]) we obtain the dynamical structure factor $\Phi_q(t)=\langle h_q(0)h_{-q}(t)\rangle_S$, using the same idea of a self-consistent expansion. Here too, the average $\langle\cdots\rangle_S$ denotes steady-state averaging, where $h_q(0)$ is measured in steady state at time $t=0$ and then $h_{-q}(t)$ is measured at some later time t (also in steady state). The dynamical structure factor $\Phi_q(t)$ normalized by $\phi_q=\Phi_q(0)$ (i.e., the static structure factor) is thus a measure of the persistence in steady state of disturbances with wave vector $\vec{q}$. Because the noise is correlated in time, we cannot use the Fokker-Planck approach, but as seen in Refs. [22,23] such a Langevin approach lends itself as an alternative to the Fokker-Planck approach even when the latter is available.

Our starting point is the field equation for $h_{q\omega}$ [the Fourier transform in time and space of $h(\vec{r},t)$] obtained by Fourier transforming Eq. (1)

$$i\omega h_{q\omega} + \nu_q h_{q\omega} + \sum_{\ell,\sigma,m,\tau} C_{q\ell m} h_{\ell\sigma} h_{m\tau} = \eta_{q\omega}, \tag{22}$$

where $\nu_q=\nu q^2$, $C_{q\ell m}=(1/\sqrt{T})(1/\sqrt{\Omega})\vec{\ell}\cdot\vec{m}\delta_{q,\ell+m}\delta_{\omega,\sigma+\tau}$, T being an assumed periodicity in time to be taken eventually to infinity, Ω is the volume of the system (to be taken to infinity as well), and the noise correlations are $\langle \eta_{q\omega}\eta_{-q-\omega}\rangle =2D^0(q)\omega^{-2\phi}$. [Note, that in the $d+1$ dimensional space (including time), the noise is quenched disorder!] In the Chapman-Enskog spirit (as done in Refs. [21,22]) the equation is written in the form

$$[(i\omega+\omega_q)h_{q\omega} - \eta^0_{q\omega}] + \lambda\left[\sum_{\ell,\sigma,m,\tau} C_{q\ell m}h_{\ell\sigma}h_{m\tau} - \eta^1_{q\omega}\right] + \lambda^2[(\nu_q-\omega_q)h_{q\omega}] = 0, \tag{23}$$

where λ is going to be taken as 1 but is used at present as an indicator to show the construction of the perturbation expansion as an expansion in λ. The noise is split into two terms $\eta_{q\omega}=\eta^0_{q\omega}+\eta^1_{q\omega}$ such that $\langle \eta^0_{q\omega}\eta^0_{-q-\omega}\rangle=D_{q\omega}$ and the correct $\Phi_{q\omega}$ [i.e., the Fourier transform in time of the "dynamical structure factor" $\Phi_q(t)$] is given by $\Phi_{q\omega}=D_{q\omega}/(\omega^2+\omega_q^2)$. This choice implies that ignoring the λ and λ^2 terms in Eq. (23), we still obtain from a linear equation the correct $\Phi_{q\omega}$. In contrast to the case of short-range correlated noise where ω_q is defined by Eq. (19), we must employ here a more general definition. The reason is that the power law found to describe the tail of $\Phi_q(t)$ for long times renders the expression on the right-hand side of Eq. (19) infinite. Therefore, our definition of ω_q is based on the assumption of a scaling form of $\Phi_q(t)$ — namely

$$\Phi_q(t) = \phi_q f(\omega_q t). \tag{24}$$

[It can be easily verified that the dynamical structure factor of the linear theory given by Eq. (16) indeed obeys this scaling law.] The "decay rate" ω_q is defined as that parameter that will make Eq. (24) a good approximation for small q's and over the whole time range.

Now, Eq. (23) enables to obtain $h_{q\omega}$ explicitly to second order in λ. The expression for $h_{q\omega}$ is multiplied into its complex conjugate and only terms up to second order in λ are retained. At the end the expressions are averaged over the noise $\eta_{q\omega}$ and we get

$$(\omega^2+\omega_q^2)\Phi_{q\omega} = D_{q\omega} + 2\lambda^2 \sum_{\ell,m,\sigma,\tau} C_{q\ell m}C_{q-\ell-m}\Phi_{\ell\sigma}\Phi_{m\tau} + \lambda^2[2D^0(q)\omega^{-2\phi} - D_{q\omega}] - 2\lambda^2(\nu_q-\omega_q)\omega_q\Phi_{q\omega} + 4\lambda^2\sum_{\ell,m,\sigma,\tau} C_{q\ell m}C_{\ell q-m}\Phi_{q\omega}\Phi_{m\tau}\frac{-i\omega+\omega_q}{i\sigma+\omega_\ell} + 4\lambda^2\sum_{\ell,m,\sigma,\tau} C_{q-\ell-m}C_{\ell-qm}\Phi_{m\tau}\Phi_{q\omega}\times\frac{i\omega+\omega_q}{-i\sigma+\omega_\ell}. \tag{25}$$

Now λ is set to be 1. The result is an equation of the form $\Phi_{q\omega}=\Phi_{q\omega}+e\{\Phi_{\ell\sigma}\}$. Equating $e\{\Phi_{\ell\sigma}\}$ to zero yields

$$[\omega^2+\omega_q^2+2(\nu_q-\omega_q)\omega_q]\Phi_{q\omega} - 2\sum_{\ell,\sigma,m,\tau}|C_{q\ell m}|^2\Phi_{\ell\sigma}\Phi_{m\tau} + 4\sum_{\ell,\sigma,m,\tau}C_{q\ell m}C_{\ell qm}\Phi_{q\omega}\Phi_{m\tau}\left[\frac{-i\omega+\omega_q}{i\sigma+\omega_\ell}+\frac{i\omega+\omega_q}{-i\sigma+\omega_\ell}\right] = 2D^0(q)\omega^{-2\phi}. \tag{26}$$

We divide the last equation by $(\omega^2+\omega_q^2)$ and using the definition $C_{q\ell m}=A_{\ell,q-\ell}\delta_{q,\ell+m}/\sqrt{\Omega T}$ (as well as letting Ω and T tend to infinity) we obtain

$$\left[1+2\frac{(\nu_q-\omega_q)\omega_q}{\omega^2+\omega_q^2}\right]\Phi_{q\omega} - 2\int\frac{d^d\ell}{(2\pi)^d}\frac{d\sigma}{2\pi}\frac{|A_{\ell,q-\ell}|^2\Phi_{\ell\sigma}\Phi_{q-\ell,\omega-\sigma}}{\omega^2+\omega_q^2} + 4\int\frac{d^d\ell}{(2\pi)^d}\frac{d\sigma}{2\pi}A_{\ell,q-\ell}A_{q,q-\ell}\left[\frac{\Phi_{q\omega}\Phi_{q-\ell,\omega-\sigma}}{[i\omega+\omega_q][i\sigma+\omega_\ell]} + \frac{\Phi_{q\omega}\Phi_{q-\ell,\omega-\sigma}}{[-i\sigma+\omega_\ell][-i\omega+\omega_q]}\right] = \frac{\omega^{-2\phi}}{\omega^2+\omega_q^2}2D^0(q). \tag{27}$$

The last equation is the basic equation for our following discussion. We consider first the small ω behavior (more specifically $\omega/\omega_q \ll 1$) that corresponds to the long-time decay of the time-dependent structure factor.

A. Long-time decay of the structure factor

The first small ω simplification is obtained by neglecting ω/ω_q. This yields

$$\left[1+2\frac{(\nu_q-\omega_q)\omega_q}{\omega_q^2}\right]\Phi_{q\omega} - 2\frac{1}{\omega_q^2}\int\frac{d^d\ell}{(2\pi)^d}\frac{d\sigma}{2\pi}|A_{\ell,q-\ell}|^2\Phi_{\ell\sigma}\Phi_{q-\ell,\omega-\sigma} + 8\frac{\Phi_{q\omega}}{\omega_q}\int\frac{d^d\ell}{(2\pi)^d}\frac{d\sigma}{2\pi}A_{\ell,q-\ell}A_{q,q-\ell}\frac{\omega_\ell}{\sigma^2+\omega_\ell^2}\Phi_{q-\ell,\omega-\sigma} = \frac{2D^0(q)}{\omega_q^2}\omega^{-2\phi}. \tag{28}$$

Fourier transforming back from frequency domain ω to real time t, we obtain

$$\left[1+2\frac{(\nu_q-\omega_q)\omega_q}{\omega_q^2}\right]\Phi_q(t) - \frac{2}{\omega_q^2}\int\frac{d^d\ell}{(2\pi)^d}|A_{\ell,q-\ell}|^2\Phi_\ell(t)\Phi_{q-\ell}(t) + \frac{8}{\omega_q}\int\frac{d^d\ell}{(2\pi)^d}A_{\ell,q-\ell}A_{q,q-\ell}\times\int_{-\infty}^{\infty}dt'\,e^{-\omega_\ell|t'|}\Phi_{q-\ell}(t')\Phi_q(t-t') = \frac{D^0(q)}{\Gamma(2\phi)\cos(\pi\phi)\omega_q^2}t^{-(1-2\phi)}, \tag{29}$$

where on the right-hand side we have written only the leading large-t behavior.

This result suggests that in the long-time limit, the time-dependent two-point function has an algebraic decay of the general form

$$\Phi_q(t) \sim A^\infty\phi_q(\omega_q t)^{-\gamma}, \tag{30}$$

where A^∞ is a numerical constant, ϕ_q is the steady-state two-point function, and γ is an exponent that will be determined later.

Equipped with the last result we can see that the first integral on the left-hand side of Eq. (29) is negligible compared to the other terms on that side in the long-time limit. The reason is that this integral decays as $t^{-2\gamma}$, while the other terms decay as $t^{-\gamma}$, making that integral subdominant for large t's.

Next, using this simplification as well as the scaling form (24), we analyze Eq. (29) for small q's (i.e., in the large scale limit) in the spirit of Refs. [9,25]. In order to achieve that, we break up the integral into the sum of two contributions corresponding to domains of $\vec{\ell}$ integration, with high and low momentum. When performing this under the assumption of long times (i.e., $\omega_q t \gg 1$) we obtain the following equation:

$$\frac{2\nu_q-\omega_q}{\omega_q}\phi_q f(\omega_q t)-\frac{8}{(2\pi)^d}\frac{\phi_q}{\omega_q}f(\omega_q t)\Bigg[\hat{D}_1 q^2 + \int^{q_0} d^d\ell A_{\ell,q-\ell}A_{q,\ell-q}\frac{\phi_{q-\ell}}{\omega_{q-\ell}}F_1\left(\{f\},\frac{\omega_\ell}{\omega_{q-\ell}}\right)\Bigg] = \frac{D^0(q)(\omega_q)^{-1-2\phi}}{A^\infty\Gamma(2\phi)\cos(\pi\phi)}(\omega_q t)^{2\phi-1}, \quad (31)$$

where f, inside the curly brackets, is just the scaling function, q_0 is the upper cut-off of the small $|\vec{\ell}|$ region, and $\hat{D}_1$ is a constant that comes from the contribution of the large $|\vec{\ell}|$ region of the first integral (see Ref. [25] Sec. VI, where such an estimation of the contribution of large momenta is also employed). In addition, we used the following notation for the integral F_1:

$$F_1\left(\{f\},\frac{\omega_\ell}{\omega_{q-\ell}}\right)=\int_0^\infty e^{-(\omega_\ell/\omega_{q-\ell})x}f(x)dx. \quad (32)$$

We conclude that $\Phi_q(t)\sim A^\infty\phi_q(\omega_q t)^{-(1-2\phi)}$ (i.e., $\gamma=1-2\phi$). And Eq. (31) can be rewritten as a time-independent equation relating the ϕ's and the ω's

$$A^\infty\frac{\phi_q}{\omega_q}\Bigg[2\nu_q+D_1q^2-\omega_q + \frac{8}{(2\pi)^d}\int^{q_0} d^d\ell A_{\ell,q-\ell}A_{q,q-\ell}\frac{\phi_{q-\ell}}{\omega_{q-\ell}}F_1\left(\{f\},\frac{\omega_\ell}{\omega_{q-\ell}}\right)\Bigg] = \frac{D^0(q)(\omega_q)^{-1-2\phi}}{\Gamma(2\phi)\cos(\pi\phi)}. \quad (33)$$

It is interesting to compare the result above for $\Phi_q(t)$ with the decay in the case where the noise is not correlated in time (namely, when $\phi=0$) [22,24]. In that case the long-time behavior of $\Phi_q(t)$ is given by

$$\Phi_q(t)\propto(\omega_q t)^{(d-1)/2z}\exp[-(\omega_q t)^{(1/z)}] \quad (34)$$

(i.e., a stretched exponential).

The limit as ϕ tends to zero of $\Phi_q(t)$ should yield a short-range decay. The expression in Eq. (30) (given that $\gamma=2\phi-1$) tends to a function that scales as t^{-1}. This should be viewed as a function that scales as $\delta(t)$ at large t's, or a short-range function. Actually, a direct inspection on the right-hand side of Eq. (29) recovers this. Since the denominator of the right-hand side contains the Γ function, the whole expression vanish as ϕ tends to zero. Checking more carefully, for $\phi=0$ the right-hand side of Eq. (29) is proportional to $\exp[-\omega_q t]$. If we try now a solution $\Phi_q(t)\propto\exp[-\omega_q t]$ we find that it does not work. The reason for that and how to obtain the correct asymptotic behavior [Eq. (34)] is detailed in Ref. [22].

To complete the picture given so far, that is, after finding the power law that governs the structure factor, we need to know the steady-state structure factor ϕ_q and its associated "decay rate" ω_q that will characterize the short-time decay as well.

B. Steady-state properties

In this part we obtain another equation, which together with Eq. (33) forms a complete set of coupled equations that will yield the small q dependence of ϕ_q and ω_q. In order to achieve this we would like to discuss Eq. (27) in the limit of short times as well. Here, it is more convenient to treat the time-dependent equation directly, so we Fourier transform, Eq. (27), to yield

$$\Phi_q(t)+(\nu_q-\omega_q)\int_{-\infty}^{\infty}dt' e^{-\omega_q|t'|}\Phi_q(t-t') - \frac{2}{(2\pi)^d}\int d^d\ell|A_{\ell,q-\ell}|^2\int_{-\infty}^{\infty}dt'\frac{e^{-\omega_q|t'|}}{2\omega_q} \times\Phi_{q-\ell}(t-t')\Phi_\ell(t-t') + \frac{4}{(2\pi)^d}\int d^d\ell A_{\ell,q-\ell}A_{q,q-\ell}\Bigg\{\int_0^\infty dt'\int_0^\infty dt'' e^{-\omega_\ell t'-\omega_q t''} \times\Phi_{q-\ell}(t')\Phi_q(t-t'-t'') + \int_{-\infty}^0 dt'\int_{-\infty}^0 dt'' e^{\omega_\ell t'+\omega_q t''}\Phi_{q-\ell}(t')\Phi_q(t-t'-t'')\Bigg\} = \frac{D^0(q)}{\pi}\int_{-\infty}^{\infty}\frac{\omega^{-2\phi}e^{i\omega t}}{\omega^2+\omega_q^2}d\omega. \quad (35)$$

Setting $t=0$ and following the same steps described above for long times (i.e., breaking the $\vec{\ell}$ integration into large and small $|\vec{\ell}|$ regions, and discussing the small q behavior of each) gives the following short-time evaluation of Eq. (35)

$$\nu_q\phi_q+\frac{8}{(2\pi)^d}\phi_q\Bigg[\hat{E}_1q^2 + \int^{q_0}d^d\ell\frac{A_{\ell,q-\ell}A_{q,q-\ell}}{\omega_\ell}\phi_{q-\ell}F_2\left(\{f\},\frac{\omega_{q-\ell}}{\omega_\ell},\frac{\omega_q}{\omega_\ell}\right)\Bigg] - \frac{2}{(2\pi)^d}\Bigg[\frac{1}{\omega_q}\int^{q_0}d^d\ell|A_{\ell,q-\ell}|^2\phi_{q-\ell}\phi_\ell F_3\left(\{f\},\frac{\omega_{q-\ell}}{\omega_q},\frac{\omega_\ell}{\omega_q}\right) + \hat{E}_2+\frac{\hat{E}_3}{\omega_q^{4\phi-1}}\Bigg] = \frac{D^0(q)}{\cos(\pi\phi)}(\omega_q)^{-2\phi}, \quad (36)$$

where as before $\hat{E}_1$, $\hat{E}_2$, and $\hat{E}_3$ are (renormalization) constants. In addition, we used the following notations:

$$F_2\left(\{f\},\frac{\omega_{q-\ell}}{\omega_\ell},\frac{\omega_q}{\omega_\ell}\right)=\int_0^\infty dx\int_0^\infty dy e^{-x-y} \times f\left(\frac{\omega_{q-\ell}}{\omega_\ell}x\right)f\left(\frac{\omega_q}{\omega_\ell}x+y\right), \quad (37)$$

and

$$F_3\left(\{f\},\frac{\omega_{q-\ell}}{\omega_q},\frac{\omega_\ell}{\omega_q}\right)=\int_0^\infty e^{-x}f\left(\frac{\omega_{q-\ell}}{\omega_q}x\right)f\left(\frac{\omega_\ell}{\omega_q}x\right)dx. \quad (38)$$

Up to this point we obtained two coupled equations for ϕ_q and ω_q. Note that the equations above depend on the (unknown) functional form of the scaling function f. We will proceed now as far as possible without specifying that form, to obtain many general results about the exponents. In the actual numerical calculation of the exponents we will resort to an approximate form of f to be described later.

We would like now to solve Eqs. (33) and (36) in the limit of small q's. For convenience we rewrite these equations using the explicit form of $A_{\ell,m}$ as

$$\phi_q[(2\nu + D_1)q^2 - \omega_q + J^<(q)] = \frac{D^0(q)(\omega_q)^{-2\phi}}{A^\infty \cos(\pi\phi)\Gamma(2\phi)}, \tag{39}$$

and

$$(\nu + E_1)q^2\phi_q + I_1^<(q)\phi_q - I_2^<(q) - E_2 - \frac{E_3}{\omega_q^{4\phi-1}} = \frac{D^0(q)(\omega_q)^{-2\phi}}{\cos(\pi\phi)}, \tag{40}$$

where

$$J^<(q) = \frac{8\lambda^2}{(2\pi)^d}\int^{q_0} d^d\ell \frac{[\vec{\ell}\cdot(\vec{q}-\vec{\ell})][\vec{q}\cdot(\vec{q}-\vec{\ell})]}{\omega_{q-\ell}}\phi_{q-\ell} \times F_1\left(\{f\}, \frac{\omega_\ell}{\omega_{q-\ell}}\right), \tag{41}$$

$$I_1^<(q) = \frac{8\lambda^2}{(2\pi)^d}\int^{q_0} d^d\ell \frac{[\vec{\ell}\cdot(\vec{q}-\vec{\ell})][\vec{q}\cdot(\vec{q}-\vec{\ell})]}{\omega_\ell} \times \phi_{q-\ell}F_2\left(\{f\}, \frac{\omega_{q-\ell}}{\omega_\ell}, \frac{\omega_q}{\omega_\ell}\right), \tag{42}$$

and

$$I_2^<(q) = \frac{2\lambda^2}{(2\pi)^d}\int^{q_0} d^d\ell \frac{[\vec{\ell}\cdot(\vec{q}-\vec{\ell})]^2}{\omega_q}\phi_{q-\ell}\phi_\ell F_3\left(\{f\}, \frac{\omega_{q-\ell}}{\omega_\ell}, \frac{\omega_q}{\omega_\ell}\right). \tag{43}$$

Note that the integrals in Eqs. (41)–(43) are cut by q_0. q_0 is chosen in such a way that below it ϕ_q and ω_q are expected to be power laws in q,

$$\phi_q = Aq^{-\Gamma}, \tag{44}$$

and

$$\omega_q = Bq^z, \tag{45}$$

where z is the dynamic exponent, and Γ is related to the roughness exponent α by

$$\alpha = (\Gamma - d)/2. \tag{46}$$

As mentioned above, the integrals in Eqs. (41)–(43) are cut by q_0, and therefore we can readily use these power laws inside the integrals. Using the power laws we can also rewrite Eqs. (39) and (40) as

$$Aq^{-\Gamma}[(2\nu + D_1)q^2 - Bq^z + J^<(q)] = \frac{D^0(q)(Bq^z)^{-2\phi}}{A^\infty\cos(\pi\phi)\Gamma(2\phi)}, \tag{47}$$

and

$$A(\nu + E_1)q^{2-\Gamma} + I_1^<(q)Aq^{-\Gamma} - I_2^<(q) - E_2 = \frac{D^0(q)(Bq^z)^{-2\phi}}{\cos(\pi\phi)}, \tag{48}$$

where we have neglected the E_3 term in Eq. (40) as it is negligible compared to the left-hand side in the limit of small q's (since $\phi < 1/2$).

It is interesting to note that these equations are a nontrivial generalization of the SCE method developed in Refs. [25,26]. More specifically, if we take the limit of $\phi \to 0$, and plug in $f(u) = e^{-u}$, which is the scaling function of the linear theory when $\phi = 0$, both equations [i.e., Eqs. (47) and (48)] reduce to the equations obtained using SCE. It is a surprise to find this similarity because the self-consistent expansion was originally derived using the Fokker-Planck equation associated with the Langevin-like KPZ equation, while the derivation given here deals directly with the Langevin form. Once we realized this surprising similarity, it is only natural to follow the asymptotic solution that is used in the well-established SCE literature, and is detailed for example in Ref. [25].

V. DETAILED ASYMPTOTIC SOLUTION

As mentioned above, in performing the asymptotic solution of the self-consistent equations, we follow previous work. We also focus here, for simplicity, on the case of noise without spatial correlations [i.e., $D^0(q) = D_0$]. However, Eqs. (47) and (48) are valid for any spatial correlations of the noise [i.e., for any $D^0(q)$], and the more general case is postponed to the following section.

The first step in the asymptotic solution is to evaluate the integrals $I_1^<(q)$, $I_2^<(q)$, and $J^<(q)$ using the power laws given in Eqs. (44) and (45)

$$I_1^<(q), J^<(q) \propto \begin{cases} q^2 & \text{for } d + 2 - \Gamma - z > 0 \\ q^{d+2-\Gamma-z} & \text{for } d + 2 - \Gamma - z < 0, \end{cases} \tag{49}$$

$$I_2^<(q) \propto \begin{cases} \text{const} & \text{for } d + 4 - 2\Gamma - 2z(1-2\phi) > 0 \\ q^{d+4-2\Gamma-z} & \text{for } d + 4 - 2\Gamma - 2z(1-2\phi) < 0. \end{cases} \tag{50}$$

We consider now the upper-right quadrant of the (Γ, z) plane, where a solution may be expected. The lines $d+2-\Gamma-z=0$ and $d+4-2\Gamma-2z(1-2\phi)=0$ divide the quadrant into four sectors. We investigate next each sector separately to decide whether a solution of the Eqs. (47) and (48) can exist there or not (in the limit of small q's).

Sector α is defined by $d+2-\Gamma-z>0$ and $d+4-2\Gamma-2z(1-2\phi)>0$. In this sector Eqs. (47) and (48) reduce to

$$Aq^{-\Gamma}[(2\nu+D_1+D_2)q^2-Bq^z]=\frac{D_0B^{-2\phi}}{A^{\infty}\cos(\pi\phi)\Gamma(2\phi)}q^{-2\phi z}, \tag{51}$$

and

$$A(\nu+E_1+E_4)q^{2-\Gamma}-E_5-E_2=\frac{D_0B^{-2\phi}}{\cos(\pi\phi)}q^{-2\phi z}. \tag{52}$$

First, the possibility that $2-\Gamma>-2z\phi$ can be ruled out immediately, because B is positive, so that Eq. (51) cannot be balanced in leading order (in powers of q). If $2-\Gamma<-2z\phi$ then the right-hand side of Eq. (51) is negligible compared to the left-hand side, so that the leading order equations are identical to those obtained for the white-noise KPZ problem and thus the standard KPZ results from Refs. [9,25] are restored. Therefore, we get $\Gamma=2$ and $z=2$. Since $\Gamma=2$ and z must be positive, the condition $2-\Gamma<-2z\phi$ can be met only for $\phi\leqslant 0$. Namely, for the case of noise anticorrelated in time [27]. Such a solution holds only for $d>4(1-2\phi)$.

The other relevant option is $2-\Gamma=-2z\phi$. This implies that ϕ must be positive, $\Gamma>2$ and $z\geqslant 2$. There is now an interesting difference between the case $z>2$ and $z=2$. For $z>2$, the leading order terms in Eqs. (51) and (52) lead to two linear homogeneous equations in the quantities A and $B^{-2\phi}$. This implies that in order to have a physical solution with $A,B>0$, we must have the determinant of the coefficient matrix vanish, namely,

$$\Gamma(2\phi)A^{\infty}(2\nu+D_1+D_2)=(\nu+E_1+E_4). \tag{53}$$

Since the quantities D_1,D_2,E_1,E_2 depend on the behavior of ϕ_ℓ and ω_ℓ for $\ell>q_0$, on the total upper cutoff etc., it is difficult to envisage that Eq. (53) can be fulfilled other than accidentally for nongeneric values of the parameters of the system. The case with $z=2$ is different. The two equations for the coefficients A and B have now an additional term—AB on the right-hand side of the first equation. This enables now a generic solution for the coefficients. In that case $\Gamma=2+4\phi$. Considering the defining conditions for the sector we find that within the sector such a solution is possible only for $d>4$.

Sector β is defined by $d+2-\Gamma-z>0$ and $d+4-2\Gamma-2z(1-2\phi)<0$. In this sector Eq. (52) is replaced by

$$A(\nu+E_1+E_4)q^{2-\Gamma}-E_6q^{d+4-2\Gamma-z}-E_2=\frac{D_0B^{-2\phi}}{\cos(\pi\phi)}q^{-2\phi z}, \tag{54}$$

while Eq. (51) remains intact.

The analysis for the possibility $2-\Gamma=-2z\phi$ in sector β is similar to the above analysis for sector α. The only difference is that due to the different defining conditions of the sector, such a solution with $z=2$ and $\Gamma=2+4\phi$ holds within the sector for $2+4\phi<d<4$.

Combining the results for sectors α and β, we see that for $\phi\leqslant 0$ the noise term is irrelevant, and the critical exponents that describe the EW problem with uncorrelated noise (namely $\Gamma=z=2$) are restored. This option is possible if $d>2$. In addition, for $\phi>0$ we get the new solution $z=2$ and $\Gamma=2+4\phi$ that is just the solution obtained for the EW equation with temporally correlated noise [see Eq. (13) above]. Following the above discussion it is realized that this solution is possible only for $d>2+4\phi$. Therefore, the lower critical dimension in this problem is $d_c=2+4\phi$ (provided $\phi>0$, otherwise $d_c=2$ as mentioned above).

Sector γ is defined by $d+2-\Gamma-z<0$ and $d+4-2\Gamma-2z(1-2\phi)>0$. In this sector Eq. (52) is replaced by

$$A(\nu+E_1)q^{2-\Gamma}+AE_7q^{d+4-2\Gamma-z}-E_5-E_2=\frac{D_0B^{-2\phi}}{\cos(\pi\phi)}q^{-2\phi z}. \tag{55}$$

First, the two defining conditions of this sector imply that the first term on the left-hand side is negligible compared to the second term, and the second term is negligible compared to the term on the right-hand side of the equation. Therefore, looking at the simplified equation, we must conclude that $\phi=0$ and $-E_5-E_2=D_0B^{-2\phi}/\cos(\pi\phi)$. However, this is impossible because the left-hand side is negative definite.

Sector δ is defined by $d+2-\Gamma-z<0$ and $d+4-2\Gamma-2z(1-2\phi)<0$. In this sector Eqs. (47) and (48) take the form

$$Aq^{-\Gamma}\left[(2\nu+D_1)q^2-Bq^z+\frac{8\lambda^2}{(2\pi)^d}\frac{A}{B}q^{d+4-\Gamma-z}G(\{f\},\Gamma,z)\right]=\frac{D_0B^{-2\phi}}{A^{\infty}\cos(\pi\phi)\Gamma(2\phi)}q^{-2\phi z} \tag{56}$$

and

$$A(\nu+E_1)q^{2-\Gamma}-\frac{2\lambda^2}{(2\pi)^d}\frac{A^2}{B}q^{d+4-2\Gamma-z}F(\{f\},\Gamma,z)-E_2=\frac{D_0B^{-2\phi}}{\cos(\pi\phi)}q^{-2\phi z}, \tag{57}$$

where $G(\{f\},\Gamma,z)$ is given by

$$G(\{f\},\Gamma,z)=\int d^dt\frac{[\vec{t}\cdot(\hat{e}-\vec{t})][\hat{e}\cdot(\hat{e}-\vec{t})]}{|\hat{e}-\vec{t}|^z}\times|\hat{e}-\vec{t}|^{-\Gamma}F_1\left(\{f\},\frac{t^z}{|\hat{e}-\vec{t}|^z}\right), \tag{58}$$

and $F(\{f\},\Gamma,z)$ given by

$$F(\{f\},\Gamma,z)=-4\int d^dt\frac{[\vec{t}\cdot(\hat{e}-\vec{t})][\hat{e}\cdot(\hat{e}-\vec{t})]}{t^z}\times|\hat{e}-\vec{t}|^{-\Gamma}F_2\left(\{f\},\frac{|\hat{e}-\vec{t}|^z}{t^z},\frac{1}{t^z}\right)+\int d^dt[\vec{t}\cdot(\hat{e}-\vec{t})]^2\times|\hat{e}-\vec{t}|^{-\Gamma}t^{-\Gamma}F_3(\{f\},|\hat{e}-\vec{t}|^z,t^z). \tag{59}$$

$\hat{e}$ is a unit vector in an arbitrary direction, and the $\vec{t}$ integration is over all d-dimensional space.

From the defining conditions of this sector, it is possible to neglect the q^2 term in the brackets on the left-hand side of Eq. (56) compared to the third $q^{d+4-\Gamma-z}$ term (since $d+4-\Gamma-z<2$). In addition, it is also possible to neglect the $q^{2-\Gamma}$

term on the left-hand side of Eq. (57) compared to the second $q^{d+4-2\Gamma-z}$ term (since $d+4-2\Gamma-z<2-\Gamma$). It is easy to see that for $\phi<0$ all the usual KPZ results are trivially retained, as the right-hand side is irrelevant then. Therefore, we focus on the case $\phi>0$ and thus the constant term on Eq. (57) can be neglected compared to the noise term on the right-hand side.

Now there are two options: First, when $\Gamma>z(1+2\phi)$ then the right-hand side of both Eqs. (56) and (57) is negligible compared to the left-hand side. In that case the critical exponents are determined by a combination of the scaling relation $d+4-\Gamma-2z=0$, and the equation

$$F[\{f\},\Gamma,z(\Gamma)]=0, \tag{60}$$

[where F is given by Eq. (59) above]. We denote the solutions of the transcendental equation by $\Gamma_\phi(d)$ (since the exponent Γ is dependent on the spatial dimension d and on ϕ). For example, in one dimension, and for $\phi=0$ it can be shown analytically that $\Gamma_0(1)=2$ and in two dimensions a numerical solution of the equation (again for $\phi=0$) yields $\Gamma_0(2)=2.59$ (see Refs. [9,25]). A discussion for general ϕ's will be given below. Still, we must remember that a solution here is obtained by requiring $\Gamma>z(1+2\phi)$. This yields a necessary condition for the existence of such a solution, $\phi<[3\Gamma_\phi(d)-d-4]/2[d+4-\Gamma_\phi(d)]$.

The second option in this sector is $\Gamma=z(1+2\phi)$ [the possibility $\Gamma<z(1+2\phi)$ is irrelevant because one cannot balance the equations and still be consistent with the defining conditions of sector δ in that situation]. Then in order to balance Eqs. (56) and (57) we must also have $d+4-2\Gamma-z=-2z\phi$. This leads to the new solution $z=(d+4)/(3+2\phi)$ and $\Gamma=(d+4)(1+2\phi)/(3+2\phi)$. However, this solution is valid only if Eq. (60) does not yield exponents (namely, Γ_ϕ and z_ϕ) that make the $q^{d+4-2\Gamma-z}$ term in Eq. (57) dominant. Not surprisingly, this requirement translates into the condition $\phi>[3\Gamma_\phi(d)-d-4]/2[d+4-\Gamma_\phi(d)]$—implying either a smooth transition between the two types of solutions or a complete domination of the first option [i.e., $\Gamma_\phi(d)$]. Actually, the existence of this new solution also requires $F[\{f\},\Gamma,z(\Gamma)]<0$. This requirement turns out to be the same as $\Gamma_\phi(d)<\Gamma_{new}=(d+4)(1+2\phi)/(3+2\phi)$, so that this extra requirement is fulfilled automatically since $\Gamma=z(1+2\phi)$.

To summarize the results of sector δ, we found two possible strong-coupling solutions. The first solution is obtained from Eq. (60), and its scaling exponents are denoted by $\Gamma_\phi(d)$ and $z_\phi(d)$ (this solution reduces to the standard KPZ results when $\phi=0$). The second solution is given by the explicit expressions $\Gamma=(d+4)(1+2\phi)/(3+2\phi)$, and $z=(d+4)/(3+2\phi)$. Then, in a given dimension d and for a given ϕ, the actual strong-coupling exponents of the KPZ problem with temporally correlated noise are just $\Gamma=\max\{\Gamma_\phi(d),(d+4)(1+2\phi)/(3+2\phi)\}$ and its corresponding z. Thus, the transition between the two solutions as a function of ϕ (if such a transition exists) is continuous. However, it should be emphasized that for a specific ϕ one of these solutions dominates so there is no phase transition between them.

Based on the results of the linear theory (13) and the numerical simulation [17] we expect the exponent Γ to be a nondecreasing function of ϕ (that means that the inclusion of temporal correlations does not make the surface smoother). This implies that $\Gamma_\phi(d)\geq\Gamma_0(d)$, and since we know $\Gamma_0(d)$ from the white-noise KPZ problem it is easy to determine a lower bound on ϕ, denoted by $\phi_c(d)$, such that the new solution $\Gamma=(d+4)(1+2\phi)/(3+2\phi)$ is not possible below it. The specific value of $\phi_c(d)$ is $\phi_c(d)=[3\Gamma_0(d)-d-4]/2[d+4-\Gamma_0(d)]$. The fact that $\Gamma_\phi(d)$ is nondecreasing as a function of ϕ implies two possible options. Either the expression $\Gamma_\phi(d)$ gives the strong-coupling solution for the whole range of $0<\phi<1/2$, or it is the solution for small ϕ and crosses over to $\Gamma=(d+4)(1+2\phi)/(3+2\phi)$. Such a crossover can occur only above $\phi_c(d)$.

We turn now to the actual evaluation of the exponents. To do that we need an ansatz for the scaling function f on which the form of the equations for Γ and z depend. Since the equations were constructed in such a way that second-order corrections to the quantities ϕ_q and ω_q vanish, we use as in Refs. [22,23], the zero-order form of the scaling function in evaluating these corrections. The scaling function in zero order is the function obtained for the corresponding linear theory, given by Eq. (17). We therefore simplify Eq. (60) using this ansatz. First, one quantity can be evaluated exactly

$$F_1(\{f_{EW}\},a)=\int_0^\infty e^{-ax}f_{EW}(x)dx=\frac{a-a^{-2\phi}}{a^2-1}. \tag{61}$$

In addition, the functional F_2 can be simplified so that it involves only one-dimensional integration (instead of double integration)

$$\begin{aligned}F_2(\{f_{EW}\},a,b)&=\int_0^\infty dx\int_0^\infty dy e^{-x-y}f_{EW}(ax)f_{EW}(bx+y)\\&=\frac{\cos(\pi\phi)}{\pi}\int_{-\infty}^{\infty}du\frac{|u|^{-2\phi}}{u^2+1}\frac{1}{1-iu}\\&\quad\times\frac{a^{1+2\phi}(1-ibu)^{-2\phi}-(1-ibu)}{[a^2-(1-ibu)^2]}\end{aligned} \tag{62}$$

(note that the integral is real, as it should be, even though the integrand is complex).

In order to illustrate the outcome of this analysis we specialize to one dimension. First, in one dimension only strong-coupling solutions are possible as the critical dimension is $2+4\phi$. Second, as mentioned above, in one dimension Eq. (60) can be solved analytically for $\phi=0$ and it gives $\Gamma_0=2$. This corresponds to a roughness exponent of $\alpha_0=1/2$ [using Eq. (46)] and to a dynamic exponent of $z_0=3/2$ [using the scaling relation $z(\Gamma)=(d+4-\Gamma)/2$]. for higher values of ϕ one has to solve Eq. (60) numerically using the ansatz of the linear theory. These results are summarized in Fig. 2 as the solid line. The figure also presents the possible second solution $\Gamma=5(1+2\phi)/(3+2\phi)$ [that corresponds to $\alpha=(1+4\phi)/(3+2\phi)$] and $z=5/(3+2\phi)$, and a continuation of Γ_0 as a dashed line. However, since the this solution is smaller than Γ_ϕ, it is practically irrelevant, since Γ_ϕ dominates the whole ϕ range.

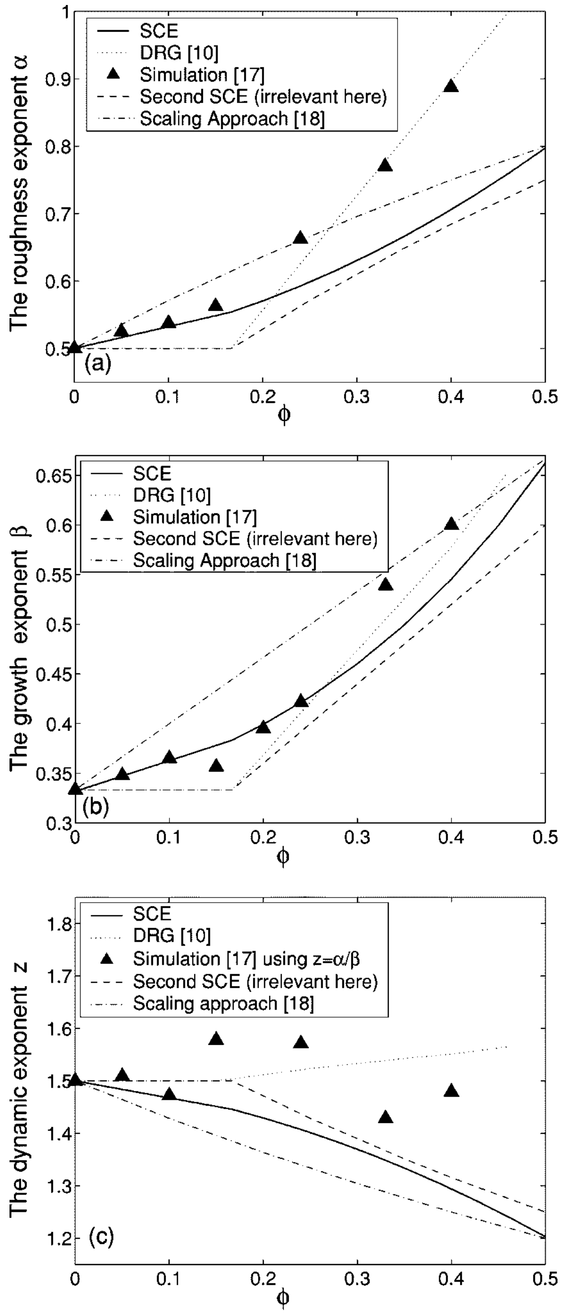

FIG. 2. (a) The roughness exponent α_ϕ, (b) the growth exponent β_ϕ, and (c) the dynamic exponent z_ϕ as a function of the exponent ϕ for decay of temporal correlations in $d=1$. Note that the dynamic exponent z was inferred for the numerical results of Ref. [17] from α and β using the scaling relation $z=\alpha/\beta$. Second, note that the DRG result is possible only up to $\phi=0.46$. Third, the dashed line shows our second possible solution (using SCE) that turns out to be irrelevant here, since it is smaller than the SCE α_ϕ for all ϕ's.

It is particularly interesting to compare this one-dimensional result to the DRG result that was presented in the Introduction [10], and plotted for convenience in Fig. 2. Generally speaking, the two methods disagree on the values of the critical exponents significantly over most of the ϕ range. Three substantial differences can be observed between the two methods. First, using the self-consistent approach we found no "threshold behavior." That is, we found a continuous variation of the scaling exponents α and z as a function of ϕ over the whole range of possible ϕ's, rather than no variation of these exponents up to a critical value of ϕ_c and a quasilinear behavior from that point on.

Second, we found a solution for the exponents for every ϕ, while the DRG approach found no solution above $\phi=0.46$ (claiming that no stable surface can grow under the condition $0.46<\phi<0.5$). Interestingly, the threshold ϕ_c (the crossover point) that was predicted using DRG in $d=1$ (namely 0.167) is the same as the lower bound we found above ($\phi_c=1/6$ for $d=1$). Our exact statement was that for $\phi>1/6$ the second strong-coupling KPZ solution becomes possible in principle (but not in practice). Therefore, the DRG result might reflect this exact statement.

Third, we found that z is a decreasing function of ϕ, while the DRG approach predicts an increasing value of z. The reason for this difference is not clear, but it might stem from the definition of the typical decay rate, which was actually defined using the scaling form (24), rather than a more "traditional" definition such as Eq. (19). The reason for using this definition is that the integral over the scaling function $f(u)$ does not converge, because of its power-law tail. Actually, in the case of the linear theory, where everything can be calculated exactly, the only possible definition is the one we used. Now, since the introduction of temporally correlated noise certainly slows down the relaxations in the system, this might have caused an artifact of increasing z, because larger z's are interpreted as longer relaxation times. However, in our approach, we do see this slowing down clearly, but it does not come from a larger dynamic exponent z in an exponential decaying scaling function, but rather from a very slowly decaying scaling function, which does not decay exponentially. Thus, this difference might reflect a better understanding of the time-dependent dynamics in such driven systems.

Comparison of the two theoretical predictions with the only numerical study of this problem (taken from Ref. [17], and plotted in Fig. 2) is inconclusive as well. It can be seen that the numerical simulation predicts a continuous variation of the scaling exponents with ϕ, rather than threshold behavior, and thus supports our predictions. In addition, when comparing the actual values of the exponents one can find good agreement between our results and the simulation for small $\phi's$ (up to $\phi\sim 1/4$). However, for larger $\phi's$ the simulation found much larger exponents, and a better numerical agreement with the DRG prediction. Still, there is some additional disagreement between the two, as the simulation found a stable surface for all ϕ's while the DRG predicts a stable surface only up to $\phi=0.46$. The last point concerns the value of the dynamic exponent z for various values of ϕ. Strictly speaking z was not given in Ref. [17], but we reconstructed it using the well known scaling relation $z=\alpha/\beta$ that is valid under very general conditions [it comes from the Family-Vicsek scaling relation [3]—see Eq. (2)]. The resulting z [see Fig. 2] is very confusing. z has no clear tendency—it just fluctuates around $z=3/2$. Therefore, the numerical data does not resolve the question of decreasing/increasing z. Certainly, future numerical results will be of great interest to clarify these issues.

The results obtained here are also in conflict with the result obtained by the scaling approach (SA) [see Eqs. (8) and (9) above, and Fig. 2]. The whole structure of the solution is different between these methods, so in spite of a formal similarity between the SA results and the second strong-coupling result, we found, (both predict nonlinear dependence on ϕ and give rational expressions for the scaling exponents, where the numerators and the nominators are linear functions of ϕ) there is a big difference between the predicted scaling exponents. Actually, the numerical study of this problem (see Ref. [17]) seems to favor our results. Therefore, the status of the SA result is questioned.

VI. GENERALIZATION TO SPATIOTEMPORALLY CORRELATED NOISE

Up to this point we discussed the relatively simple case of noise without any spatial correlations [i.e., $D^0(q)=D_0$]. However, including spatial correlations bears no principal difficulty to the analysis presented above. For example one can just replace D_0 with $D^0(q)=D_0 q^{-2\rho}$ in Eq. (51) and by following the steps from that point the scaling exponents can be easily extracted. For the sake of presenting a complete picture we briefly summarize the results obtained for this case. First, there is the weak-coupling solution, which is again just the corresponding EW result for such a noise, given by

$$z=2 \quad \text{and} \quad \Gamma = 2+2\rho+4\phi. \tag{63}$$

The weak-coupling solution is possible for $d>2+2\rho+4\phi$, so that here the lower critical dimension is $d_c=2+2\rho+4\phi$.

Second, there is the strong-coupling solution, given by

$$z=\begin{cases} z_\phi(d) & 2\rho+(1+2\phi)z_\phi(d)<\Gamma_\phi(d) \\ \dfrac{d+4-2\rho}{3+2\phi} & 2\rho+(1+2\phi)z_\phi(d)>\Gamma_\phi(d) \end{cases}$$

and

$$\Gamma=\begin{cases} \Gamma_\phi(d) & 2\rho+(1+2\phi)z_\phi(d)<\Gamma_\phi(d) \\ \dfrac{(d+4-2\rho)(1+2\phi)}{3+2\phi} & 2\rho+(1+2\phi)z_\phi(d)>\Gamma_\phi(d), \end{cases} \tag{64}$$

where as before, $\Gamma_\phi(d)$ is the solution of the Eq. (60) and $z_\phi(d)=[d+4-\Gamma_\phi(d)]/2$.

Furthermore, the method presented above is not restricted to noise terms that have separable correlators, i.e., $D(q,\omega)=D^0(q)\omega^{-2\phi}$, and can just as well deal with nonseparable correlators [that is, any functional form of $D(q,\omega)$]. In that case, the only difference would be to replace the right-hand side of Eq. (27) by the expression $D(q,\omega)/(\omega^2+\omega_q^2)$ with the required $D(q,\omega)$ inside.

In order to demonstrate this option, we discuss an interesting application of this approach to the KPZ equation with a very special kind of spatiotemporally correlated noise (this result was mentioned at the end of the introduction). This problem was previously solved by Li *et al.* [19] in the context of vortex lines in the three-dimensional *XY* model with random phase shifts, and it boils down to solving the KPZ equation in two dimensions (more specifically the $2+1$ case) with a noise term that has the following spatiotemporal correlations:

$$\langle \eta(\vec r,t)\eta(\vec r',t')\rangle = \frac{\sigma(Jm)^2}{\sqrt{(\vec r-\vec r')^2+(t-t')^2}}, \tag{65}$$

where $\vec r$ is a two-dimensional vector [i.e., $\vec r=(x,y)\in R^2$]. In order to apply the method presented above for finding the critical exponents of this model we have to Fourier transform the noise correlator first

$$\langle \eta(\vec q,\omega)\eta(\vec q',\omega')\rangle = \sigma(Jm)^2\frac{\delta^2(\vec q+\vec q')\,\delta(\omega+\omega')}{q^2+\omega^2}, \tag{66}$$

(where $\vec q$ is a two-dimensional vector) so that $D(q,\omega)=D_0/(q^2+\omega^2)$ [with $D_0\equiv\sigma(Jm)^2$]. Now, for small q's $\omega_q=Bq^z$, and we can evaluate the Fourier integral explicitly (assuming $z>1$)

$$\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{(\omega^2+\omega_q^2)(\omega^2+q^2)}d\omega \sim \frac{1}{\omega_q^3}\frac{\pi}{2}\frac{e^{-\omega_q t}}{q^2/\omega_q^2}=\frac{1}{q^{2+z}}\frac{\pi e^{-\omega_q t}}{2B}. \tag{67}$$

As mentioned above, this expression should replace the right-hand side of Eqs. (29) and (35). One can easily be convinced that this term is subdominant in the long-time limit, and therefore drops out of Eq. (33). However, in the short-time limit this term is not negligible, and therefore modifies Eq. (36) accordingly.

Following all the required steps from that point on leads to the results $\Gamma=z+2$ and $d+4-2z-\Gamma=0$, so that for the case we were interested in $(d=2)$ we get

$$z=\frac{4}{3} \quad \text{and} \quad \Gamma=\frac{10}{3}, \tag{68}$$

identical to the analytic result presented in Ref. [19] (a numerical analysis presented there also verifies this result).

VII. SUMMARY AND CONCLUSIONS

In this paper we developed a time-dependent self-consistent approach to deal with the KPZ equation driven by a temporally correlated noise. This achievement was made possible thanks to the observation that there is a time scale separation between short-time and long-time behavior of the system. More specifically, it was realized that when temporally correlated noise is present in the system, then slow relaxations of various time-dependent quantities should control the long-time behavior [in this case algebraic decay of the time-dependent correlation function $\Phi_q(t)$]. In addition, it was seen that the short-time behavior is influenced by the long-time behavior and vice versa.

To summarize the results briefly, we found that the KPZ equation with temporally correlated noise, just like the problem with white noise, has both a strong-coupling and a weak-coupling solution. The weak-coupling solution is described by the scaling exponents of the corresponding linear theory

(EW equation), and is made possible for dimensions higher than the lower critical dimension $d_c = 2 + 4\phi$ [the specific values for the exponents are given in Eq. (63)]. The strong-coupling solution, which is relevant also for low dimensions, is described by scaling exponents that are a result of a competition between two possible solutions. First, there is an extension of the classical white-noise KPZ solution denoted by $\Gamma_\phi(d)$ that is derived from an integral equation (60). The other possible solution is a new strong-coupling solution and is given in Eq. (64). The actual exponent Γ that describes the surface is the maximum between the two options. For small ϕ's $\Gamma_\phi(d)$ is the solution but it may (or may not, as in the one-dimensional case) cross over in a continuous manner to the second solution. For a detailed discussion and comparison to other methods in one dimension see Sec. V.

The problem dealt with in this paper is not a new one, yet very few results are available in the literature. Therefore, the result presented here might contribute to the clarification of this fundamental problem. Since the results of the self-consistent approach are different from those of DRG, in that no threshold behavior appears, and even more significantly different from the results of the scaling approach, it is a good idea to address this question numerically in order to decide between the competing results. A revision of the DRG approach to this problem might also help here, since, as was suspected in Sec. V, the difference between the DRG results and those derived here in one dimension might stem from a misunderstanding of the long-time behavior of the system. As was shown in the last section, there is a big advantage in using the self-consistent approach as it can be generalized in a simple way to discuss noise with arbitrary spatiotemporal correlations. Two such examples were discussed, and explicit predictions were made. Here too, future research may help to test the validity of these predictions.

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